Self-dual model coupled to bosons

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In this paper we investigate the dynamics, at the quantum level, of the self-dual field minimally coupled to bosons. In this investigation we use the Dirac bracket quantization procedure to quantize the model. Also, the relativistic invariance is tested in connection with the elastic boson-boson scattering amplitude.

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Quantum field theory in 2 + 1 dimensions has been provided as a mechanism to understand the quantum Hall effect, and superconductivity at high temperature. In addition, interesting aspects such as exotic statistics, fractionary spin, and insight on the existence of massive gauge field are present in the context of three dimensional models [1].

In (2+1) dimensions self-dual theories were originally proposed by Townsend, Pilch, and van Nieuwenhuizen [2,3]. Self-duality means that two equivalent descriptions of a model using different fields exist, as has been treated in Ref. [4] where the equivalence between the free self-dual model and the Maxwell-Chern-Simons (MCS) model was demonstrated at the semiclassical level. This equivalence was also observed in the level of the Green functions [5].

The exact equivalence between the self-dual model minimally coupled to a Dirac field and the MCS model with nonminimal magnetic coupling to fermions has been studied by Gomes *et al*; see Ref. [6]. Also, the canonical quantization of the self-dual model coupled to fermions has been studied by Girotti in Ref. [7]. In this study, one has observed that two new interaction terms arise, which are local in space-time and are nonrenormalizable by power counting. Relativistic invariance is tested in connection with the elastic fermion-fermion scattering amplitude.

In the present paper we will study the self-dual model coupled to bosons. We quantize this model using the Dirac bracket quantization procedure. In this quantization procedure, the results one gets are apparently Lorentz noninvariant. For this reason, we test Lorentz invariance of our results in connection with the elastic boson-boson scattering amplitude. As a result, we demonstrate that the combined action of the noncovariant pieces that make up the interaction in the Hamiltonian can be replaced by the minimal covariant field-current interaction.

We adopt the Heaviside-Lorentz units, and put $\hbar=c=1$. The metric tensor is $g^{\mu\nu}=\mathrm{diag}(1,-1,-1)$ and antisymmetric tensor $\epsilon^{\mu\nu\rho}$ is normalized as $\epsilon^{012}=1$. Also, we have considered $\epsilon^{ij}=\epsilon^{0ij}$.

The Lagrangian density that describes the self-dual field coupled to bosons is written as

$$\mathcal{L}=-\frac{1}{2m}\epsilon^{\mu\nu\rho}(\partial_\mu f_\nu)f_\rho+\frac{1}{2}f^\mu f_\mu+(D_\mu\phi)^*(D^\mu\phi)-M^2\phi^*\phi, \tag{1}$$

where the covariant derivative is given by $D_\mu=\partial_\mu+(ig/m)f_\mu$, f_μ is the self-dual field and ϕ is the charged scalar field.

The momenta canonically conjugated are

$$\pi_\alpha=\frac{\partial\mathcal{L}}{\partial(\partial_0 f_\alpha)}=-\frac{1}{2m}\epsilon_{0\alpha\rho}f^\rho, \tag{2}$$

$$\Pi=\frac{\partial\mathcal{L}}{\partial(\partial_0\phi)}=\partial^0\phi^*-\frac{ig}{m}f^0\phi^*, \tag{3}$$

$$\Pi^*=\frac{\partial\mathcal{L}}{\partial(\partial_0\phi^*)}=\partial^0\phi+\frac{ig}{m}f^0\phi. \tag{4}$$

The primary constraints are

$$P_0=\pi_0\approx 0, \tag{5}$$

$$P_i=\pi_i+\frac{1}{2m}\epsilon_{ij}f^j\approx 0, \tag{6}$$

where the sign of weak equality ($\approx$) is used in the sense of Dirac [8,9]. The canonical Hamiltonian density is given by

$$\mathcal{H}=\pi_i\partial^0 f^i+\Pi\partial^0\phi+\Pi^*\partial^0\phi^*-\mathcal{L}, \tag{7}$$

then,

$$\mathcal{H}=\Pi^*\Pi-\partial_k\phi^*\partial^k\phi+M^2\phi^*\phi+\frac{ig}{m}(f_k\phi^*\partial^k\phi-\partial_k\phi^* f^k\phi)-\frac{g^2}{m^2}f_kf^k\phi^*\phi-\frac{1}{2}f^\mu f_\mu+f^0\left[\frac{1}{m}\epsilon_{ij}f^0\partial^i f^j-\frac{ig}{m}(\Pi\phi-\Pi^*\phi^*)\right]. \tag{8}$$

The primary Hamiltonian is

$$H_P=\int d^2x(\mathcal{H}+U^0P_0+U^iP_i) \tag{9}$$

where U^0 and U^i are the Lagrange multipliers.

Imposing the consistency conditions to the constraint

$$\dot{P}_0=\{P_0,H_P\}_P=\{\pi_0(\vec{x}),H_P(\vec{y})\}_P\approx 0, \tag{10}$$

we find the secondary constraint

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$$\mathcal{S}=f^0-\frac{1}{m}\epsilon_{ij}\partial^i f^j+\frac{ig}{m}(\Pi\phi-\Pi^*\phi^*)\approx 0. \tag{11}$$

Imposing again the same condition of consistency to the constraints P_i and $\mathcal{S}$ we can verify that no more constraints arise. So we can determine the Lagrange multipliers and all constraints are second class. Following the Dirac bracket quantization procedure we get the commutation relation in equal time of the dynamics variables

$$[f^0(\vec{x}),f^j(\vec{y})]=i\partial_x^j\delta(\vec{x}-\vec{y}), \tag{12}$$

$$[f^k(\vec{x}),f^j(\vec{y})]=-im\epsilon^{kj}\delta(\vec{x}-\vec{y}), \tag{13}$$

$$[f^0(\vec{x}),\pi_k(\vec{y})]=-\frac{i}{2m}\epsilon_{kj}\partial_x^j\delta(\vec{x}-\vec{y}), \tag{14}$$

$$[f^j(\vec{x}),\pi_k(\vec{y})]=\frac{i}{2}g_k^j\delta(\vec{x}-\vec{y}), \tag{15}$$

$$[\pi_j(\vec{x}),\pi_k(\vec{y})]=-\frac{i}{4m}\epsilon_{jk}\delta(\vec{x}-\vec{y}), \tag{16}$$

$$[f^0(\vec{x}),\phi(\vec{y})]=-\frac{g}{m}\phi(\vec{x})\delta(\vec{x}-\vec{y}), \tag{17}$$

$$[f^0(\vec{x}),\phi^\dagger(\vec{y})]=\frac{g}{m}\phi^\dagger(\vec{x})\delta(\vec{x}-\vec{y}), \tag{18}$$

$$[\phi(\vec{x}),\Pi(\vec{y})]=i\delta(\vec{x}-\vec{y}), \tag{19}$$

$$[\phi^\dagger(\vec{x}),\Pi^\dagger(\vec{y})]=i\delta(\vec{x}-\vec{y}) \tag{20}$$

and all other commutators vanish.

The Hamiltonian that describes the quantum dynamics of the system is written as

$$H=\int d^2x\bigg[\Pi^\dagger\Pi-\partial_k\phi^\dagger\partial^k\phi+M^2\phi^\dagger\phi+\frac{ig}{m}(f^k\phi^\dagger\partial_k\phi-\partial_k\phi^\dagger f^k\phi)-\frac{g^2}{m^2}f_kf^k\phi^\dagger\phi+\frac{1}{2}f^0f^0+\frac{1}{2}f^if^i\bigg], \tag{21}$$

where we have considered the Wick order of the operators. We can simplify the Hamiltonian eliminating the operator f^0 using the condition that is described by Eq. (11) so that it takes the form

$$H^I=H_0^I+H_{int}^I, \tag{22}$$

where

$$H_0^I=\int d^2x\bigg[\frac{1}{2m^2}\epsilon^{ij}\epsilon^{kl}(\partial_if_j^I)(\partial_kf_l^I)+\frac{1}{2}f_i^If_i^I\bigg]+\int d^2x[\Pi^{I\dagger}\Pi^I-\partial_k\phi^{I\dagger}\partial^k\phi^I+M^2\phi^{I\dagger}\phi^I], \tag{23}$$

and

$$H_{int}^I=\int d^2x\bigg[\frac{ig}{m}(f^{Ik}\phi^{I\dagger}\partial_k\phi^I-\partial_k\phi^{I\dagger}f^{Ik}\phi^I)-\frac{g^2}{m^2}f_k^If^{Ik}\phi^{I\dagger}\phi^I-\frac{ig}{m^2}\epsilon^{kl}\partial_kf_l^I(\Pi^I\phi^I-\Pi^{I\dagger}\phi^{I\dagger})-\frac{g^2}{2m^2}(\Pi^I\phi^I-\Pi^{I\dagger}\phi^{I\dagger})(\Pi^I\phi^I-\Pi^{I\dagger}\phi^{I\dagger})\bigg]. \tag{24}$$

The superscript I denotes field operators belonging to the interaction picture.

The rules of commutations relations in equal times obeyed by operators of field in interaction pictures are exactly the equations (12)-(20). The motion equations that satisfy the operators ϕ^I and $\phi^{I\dagger}$ are

$$\partial_0\phi^I=i[H_0^I,\phi^I]=\Pi^\dagger, \tag{25}$$

$$\partial_0\phi^{I\dagger}=i[H_0^I,\phi^{I\dagger}]=\Pi \tag{26}$$

and the correspondent propagator of bosons $\Delta(p)$ in the momentum space is

$$\Delta(p)=\frac{i}{p^2-M^2+i\epsilon}. \tag{27}$$

The Feynman propagator of the self-dual field $f_i^I, i=1,2$ is given by

$$D_{lj}(k)=\frac{i}{k^2-m^2+i\epsilon}(-m^2g_{lj}+k_lk_j-im\epsilon_{lj}k_0)=D_{jl}(-k) \tag{28}$$

as has been obtained by Girotti [7].

Finally, the Hamiltonian of interactions described in terms of the fundamental fields is written as

$$H_{int}^I=\int d^2x\bigg[\frac{ig}{m}(f^{Ik}\phi^{I\dagger}\partial_k\phi^I-\partial_k\phi^{I\dagger}f^{Ik}\phi^I)-\frac{g^2}{m^2}f_k^If^{Ik}\phi^{I\dagger}\phi^I-\frac{ig}{m^2}\epsilon^{kl}\partial_kf_l^I(\partial_0\phi^{I\dagger}\phi^I-\phi^{I\dagger}\partial_0\phi^I)-\frac{g^2}{2m^2}(\partial_0\phi^{I\dagger}\phi^I-\phi^{I\dagger}\partial_0\phi^I)(\partial_0\phi^{I\dagger}\phi^I-\phi^{I\dagger}\partial_0\phi^I)\bigg]. \tag{29}$$

Observe that Eq. (29) contains four terms. The first term is the spatial part of the field-current interaction. The third term is the magnetic field interacting with the temporal component of the current, whereas the second term is the spatial part of the gauge-boson field interaction and the fourth term is the interaction of a kind of temporal component of currents. Unlike the case of MCS minimally coupled to bosons, these extra terms are strictly local in space-time. Also, they are nonrenormalizable by power counting.

In the next step we will verify the Lorentz invariance of the theory. To do this, we will evaluate the contribution of order g^2 to the lowest order elastic boson-boson scattering amplitude which can be grouped into four different kind of terms:

$$S^{(2)}=\sum_{\alpha=1}^{4} S_{\alpha}^{(2)}, \tag{30}$$

where

$$\begin{aligned}S_1^{(2)}=\frac{g^2}{2m^2}\int\int d^3x d^3y\langle\varphi_f|T\{:\{\phi^{\dagger}(x)\partial_j\phi(x)\\ -\partial_j\phi^{\dagger}(x)\phi(x)\}f^j(x)::\{\phi^{\dagger}(y)\partial_l\phi(y)\\ -\partial_l\phi^{\dagger}(y)\phi(y)\}f^l(y):\}|\varphi_i\rangle,\end{aligned} \tag{31}$$

$$\begin{aligned}S_2^{(2)}=-\frac{g^2}{m^3}\int\int d^3x d^3y\langle\varphi_f|T\{:\{\phi^{\dagger}(x)\partial_j\phi(x)\\ -\partial_j\phi^{\dagger}(x)\phi(x)\}f^j(x)::\{\epsilon_{il}\partial^i f^l(y)(\partial_0\phi^{\dagger}(y)\phi(y)\\ -\phi^{\dagger}(y)\partial_0\phi(y)\}:\}|\varphi_i\rangle,\end{aligned} \tag{32}$$

$$\begin{aligned}S_3^{(2)}=\frac{g^2}{2m^4}\int\int d^3x d^3y\langle\varphi_f|T\{:\{\epsilon_{kj}\partial^k f^j(x)(\partial_0\phi^{\dagger}(x)\phi(x)\\ -\phi^{\dagger}(x)\partial_0\phi(x))\}::\{\epsilon_{il}\partial^i f^l(y)\partial_0\phi^{\dagger}(y)\phi(y)\\ -\phi^{\dagger}(y)\partial_0\phi(y)\}:\}|\varphi_i\rangle,\end{aligned} \tag{33}$$

$$\begin{aligned}S_4^{(2)}=\frac{ig^2}{2m^2}\int\int d^3x d^3y\,\delta(x-y)\langle\varphi_f|T\{:\{\partial_0\phi^{\dagger}(x)\phi(x)\\ -\phi^{\dagger}(x)\partial_0\phi(x)\}::\{\partial_0\phi^{\dagger}(y)\phi(y)\\ -\phi^{\dagger}(y)\partial_0\phi(y)\}:\}|\varphi_i\rangle.\end{aligned} \tag{34}$$

Here, T is the chronological ordering operator, whereas $|\varphi_i\rangle$ and $\langle\varphi_f|$ denote the initial and final state of the reaction, respectively. For the case under analysis, both $|\varphi_i\rangle$ and $\langle\varphi_f|$ are two-boson states.

In terms of the initial (p_1,p_2) and the final momenta (p_1',p_2'), the partial amplitudes are

$$\begin{aligned}S_1^{(2)}=-g^2N_p(2\pi)^3\delta^3(p_1'+p_2'-p_1-p_2)\\ \times\left\{(p_1'+p_1)_j(p_2'+p_2)_l\frac{1}{m^2}D^{jl}(k)+p_1\leftrightarrow p_2\right\},\end{aligned} \tag{35}$$

$$\begin{aligned}S_2^{(2)}=-g^2N_p(2\pi)^3\delta^3(p_1'+p_2'-p_1-p_2)\\ \times\left\{(p_1'+p_1)_j(p_2'+p_2)_0\frac{1}{m^2}\Gamma^j(k)+(p_1'+p_1)_0(p_2'\right.\\ \left.+p_2)_j\frac{1}{m^2}\Gamma^j(-k)+p_1\leftrightarrow p_2\right\},\end{aligned} \tag{36}$$

$$\begin{aligned}S_3^{(2)}=-g^2N_p(2\pi)^3\delta^3(p_1'+p_2'-p_1-p_2)\\ \times\left\{(p_1'+p_1)_0(p_2'+p_2)_0\frac{1}{m^2}\Lambda(k)+p_1\leftrightarrow p_2\right\},\end{aligned} \tag{37}$$

$$\begin{aligned}S_4^{(2)}=-g^2N_p(2\pi)^3\delta^3(p_1'+p_2'-p_1-p_2)\\ \times\left\{(p_1'+p_1)_0(p_2'+p_2)_0\frac{i}{m^2}+p_1\leftrightarrow p_2\right\},\end{aligned} \tag{38}$$

where

$$\frac{1}{m^2}D^{jl}(k)=\frac{i}{k^2-m^2+i\epsilon}\left(-g^{jl}+\frac{k^jk^l}{m^2}-\frac{i}{m}\epsilon^{jl}k_0\right), \tag{39}$$

$$\frac{1}{m^2}\Gamma^j(k)=\frac{i}{k^2-m^2+i\epsilon}\left(\frac{i}{m}\epsilon^{jl}k_l+\frac{k^jk^0}{m^2}\right), \tag{40}$$

$$\frac{1}{m^2}\Lambda(k)=\frac{i}{k^2-m^2+i\epsilon}\left(-\frac{k^lk_l}{m^2}\right), \tag{41}$$

and

$$k\equiv(p_1'-p_1)=-(p_2'-p_2) \tag{42}$$

is the momentum transfer. Substituting Eqs. (35)–(38) into Eq. (30) we find

$$\begin{aligned}S^{(2)}=-g^2N_p(2\pi)^3\delta^3(p_1'+p_2'-p_1-p_2)\\ \times\left\{(p_1'+p_1)_\mu(p_2'+p_2)_\nu\frac{1}{m^2}D^{\mu\nu}(k)+p_1\leftrightarrow p_2\right\}\end{aligned} \tag{43}$$

where

$$\frac{1}{m^2}D^{\mu\nu}(k)=-\frac{i}{k^2-m^2+i\epsilon}\left(g^{\mu\nu}-\frac{k^\mu k^\nu}{m^2}+\frac{i}{m}\epsilon^{\mu\nu\alpha}k_\alpha\right) \tag{44}$$

is the propagator of the self-dual field f^μ. As can be seen the amplitude $S^{(2)}$ is the scalar of Lorentz. Observe, also, that the theory has passed in test of the relativistic invariance. On the other hand, in the tree approximation is allowed to replace all the noncovariant terms in H^I_{int}, Eq. (29), by the minimal covariant interaction $(g/m)J^I_\mu f^{I\mu}$, where J^I_μ is given by

$$J^I_\mu = i(\phi^{I*}\partial_\mu\phi^I - \partial_\mu\phi^{I*}\phi^I) - \frac{g}{m} f^I_\mu \phi^{I*}\phi^I. \tag{45}$$

We have observed also that the high energy behavior of the propagator in Eq. (44) is radically different from the MCS theory in the Landau gauge [10],

$$D_L^{\mu\nu}(k) = -\frac{i}{k^2 - m^2 + i\epsilon}\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} + \frac{im}{k^2}\epsilon^{\mu\nu\alpha}k_\alpha\right), \tag{46}$$

and therefore, the self-dual model coupled to bosons is a non-renormalized theory as we have noted previously by power counting.

Finally, we conclude in this paper that the self-dual model minimally coupled to bosons bears no resemblance with the renormalizable model defined by the MCS field minimally coupled to bosons. The equivalence between self-dual and MCS when coupled to bosons is under investigation.

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